Harmonic oscillator and nuclear pseudospin

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Abstract. A generalized relativistic harmonic oscillator for spin 1/2 particles is studied. The Dirac Hamiltonian contains a scalar $S$ and a vector $V$ quadratic potentials in the radial coordinate, as well as a tensor potential $U$, linear in $r$. Setting either $\Sigma = S + V$ or $\Delta = V - S$ to zero, analytical solutions for bound states are found. The eigenenergies and their nonrelativistic limits are presented and particular cases are discussed, especially the case $\Sigma = 0$, for which pseudospin symmetry is exact.

GENERALIZED DIRAC OSCILLATOR

The general Dirac Hamiltonian for scalar $S$, vector $V$ and tensorial $U$ potentials can be written as [1, 2, 3]

$$H = \alpha \cdot p + \beta m + \frac{1 + \beta}{2} \Sigma + \frac{1 - \beta}{2} \Delta + i \beta \cdot \hat{r} U$$  \hspace{1cm} (1)

where $\Delta = V - S$ and $\Sigma = V + S$. We consider here solutions of the time-independent Dirac equation $H \Psi = \epsilon \Psi$, when $V$, $S$ and $U$ are radial potentials. In this case, the general form of the spinor $\Psi$, with quantum numbers $\kappa$ and $m$, is

$$\Psi_{\kappa m}(r) = \begin{pmatrix} g_{\kappa}(r) & \varphi_{\kappa m}(\hat{r}) \\ f_{\kappa}(r) & \varphi_{-\kappa m}(\hat{r}) \end{pmatrix}.$$  \hspace{1cm} (2)

We can obtain second-order differential equations for either upper or lower component radial wave functions $g_{\kappa}$ and $f_{\kappa}$ with quadratic potentials in the radial coordinate. This can be achieved by setting $S = \pm V = \pm \frac{1}{4} m \omega_2^2 r^2$ and $U = m \omega_2 r$.

Case $\Delta = 0$

In this case one gets a second order Shrödinger-like equation for the upper radial function which has analytical bound state solutions [3, 4]. The corresponding eigenvalues are

$$\epsilon^2 - m^2 - (2\kappa - 1)m\omega_2 = \left(2n + l + \frac{3}{2}\right) \sqrt{2m(\epsilon^2 + m)\omega_2^2 + 4m^2 \omega_2^3}.$$  \hspace{1cm} (3)
The nonrelativistic limit of Eq. (3) is obtained by letting $\omega_1/m$ and $\omega_2/m$ become very small. In this limit, $\varepsilon' + m \sim 2m$ and $(\varepsilon' - m)/m = E/m$ is also very small, so that
\[
E = \left(2n + l + \frac{3}{2}\right) \sqrt{\omega_1^2 + \omega_2^2 + \left(\kappa - \frac{1}{2}\right)\omega_2}.
\] (4)

Figure 1 shows that the tensor coupling potential gives rise to a spin-orbit coupling, removing the degeneracy between states with quantum numbers $nlj$ when $\omega_2 = 0$. In this latter case, from Eq. (4), one gets the known relativistic result for the binding energy, $E = (2n + l + 3/2)|\omega_1|$, showing the case $\Delta = 0$, $U = 0$ and $\Sigma = 1/2 m \omega_1^2$ can be regarded as the most natural way to introduce the harmonic oscillator in relativistic quantum mechanics.

Case $\Sigma = 0$

When $\Sigma = 0$ we are able to get analytical bound state solutions for the lower radial function $g_\kappa$ [3, 4]. The corresponding eigenvalues are
\[
\varepsilon'^2 - m^2 + (2\kappa - 1)m\omega_2 = \left(2n + l + \frac{3}{2}\right) \sqrt{2m(\varepsilon' - m)\omega_1^2 + 4m^2\omega_2^2}.
\] (5)
The non-relativistic limit of this equation is obtained as before, yielding

$$E = \left(2\tilde{n} + \tilde{l} + \frac{3}{2}\right)\omega_2 - \left(\tilde{k} - \frac{1}{2}\right)\omega_2 + \frac{1}{4m}\left(2\tilde{n} + \tilde{l} + \frac{3}{2}\right)\omega_2^2 - \text{sign}(\omega_2)\left(\tilde{k} - \frac{1}{2}\right)\omega_2^2. \quad (6)$$

In Figure 2 and Eq. (6) $\tilde{n}$ and $\tilde{l}$ are, respectively, the principal quantum number (number of nodes) and the orbital quantum number of the lower component of the Dirac spinor. These are also referred to as pseudospin quantum numbers.

When $\omega_2 = 0$ (i.e., no tensor potential), there is no pseudospin orbital coupling so that $\tilde{l}$ is a good quantum number, as can be seen from Figure 2. In this case we have exact pseudospin symmetry, which is believed to be the symmetry underlying the degeneracy observed in levels with the same $\tilde{n}$ and $\tilde{l}$ near the Fermi sea in certain heavy nuclei [5, 6, 7]. From Eq. (6) we see that, when $\omega_2 = 0$, there is only a quadratic term in $\omega_1$, which means that pseudospin symmetry is an intrinsically relativistic theory, having no first-order terms in a $\omega_1/m$ expansion [3]. It is worthwhile to emphasize the fact that with a harmonic oscillator type of potential it is possible to have bound solutions of the Dirac equation with $\Sigma = U = 0$. This is not possible in relativistic mean-field theories for nuclei, where the scalar and vector potentials vanish as $r \rightarrow \infty$ and $\Sigma$ plays the role of a binding potential. In the present case, the $\Delta$ potential, with a harmonic oscillator shape, can provide such binding.

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